

## 5.1 – Eigenvalues and Eigenvectors

Consider the equation

$$\begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix}$$

→ form:  $A\vec{x} = \lambda\vec{x}$

↙ lambda

**Definition:** If  $A$  is an  $n \times n$  matrix, then a nonzero vector  $\mathbf{x}$  in  $R^n$  is called an **eigenvector** of  $A$  (or of the matrix operator  $T_A$ ) if  $A\mathbf{x}$  is a scalar multiple of  $\mathbf{x}$ ; that is,  $A\mathbf{x} = \lambda\mathbf{x}$  for some scalar  $\lambda$ . The scalar  $\lambda$  is called an **eigenvalue** of  $A$  (or of  $T_A$ ), and  $\mathbf{x}$  is said to be an **eigenvector corresponding to  $\lambda$** .

In the above equation, 5 is an eigenvalue of the matrix  $A$ , and  $\begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix}$  is its associated eigenvector.

Note that if  $A\vec{x} = \lambda\vec{x}$ , then  $\lambda\vec{x} - A\vec{x} = \vec{0}$ .  
 then  $(\lambda I - A)\vec{x} = \vec{0}$ . Since  $\vec{x} \neq \vec{0}$   
 we seek nontrivial solutions.

**Theorem 5.1.1** If  $A$  is an  $n \times n$  matrix, then  $\lambda$  is an eigenvalue of  $A$  if and only if it satisfies the equation  $\det(\lambda I - A) = 0$ .

If  $A$  is an  $n \times n$  matrix, then the **characteristic equation** of  $A$  is  $\det(\lambda I - A) = 0$ .

**Definition:** The **characteristic polynomial** of an  $n \times n$  matrix  $A$  results from expanding the characteristic equation of  $A$  and has the form

$$p(\lambda) = \lambda^n + c_1\lambda^{n-1} + \dots + c_n.$$

The solution space of the system  $(\lambda I - A)\mathbf{x} = \mathbf{0}$  is called the **eigenspace** of  $A$  corresponding to  $\lambda$ .

# $A - \lambda I$

#6 In each part, find the characteristic equation, the eigenvalues, and bases for the eigenspaces of the matrix.

a.  $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

b.  $\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$

c.  $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

d.  $\begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$

a)  $\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{bmatrix}$

$\det(\lambda I - A) = 0 \Rightarrow \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = 0$

Characteristic equation

$(\lambda - 2)^2 - 1 = 0 \Rightarrow (\lambda - 2)^2 = 1 \Rightarrow \lambda - 2 = \pm 1$

$\lambda = 1, 3$  eigenvalues

$\lambda_1 = 3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \rightarrow x_1 = x_2 \quad \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t$

Basis for eigenspace is  $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ .  $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 1 \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$\rightarrow$  basis is  $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ .

b)  $\begin{vmatrix} \lambda - 2 & -3 \\ 0 & \lambda - 2 \end{vmatrix} = 0 \Rightarrow (\lambda - 2)^2 = 0$

$\lambda = 2$  is an eigenvalue

of algebraic multiplicity 2

$\begin{bmatrix} 0 & -3 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \rightarrow x_2 = 0$

$x_1$  is free

$\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} t$

Basis:  $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

$\lambda = 2$  has geometric multiplicity 1

$$c) \begin{vmatrix} \lambda-2 & 0 \\ 0 & \lambda-2 \end{vmatrix} = 0 \Rightarrow (\lambda-2)^2 = 0 \Rightarrow \lambda = 2$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad x_1, x_2 \text{ free} \Rightarrow \text{basis: } \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$d) \begin{vmatrix} \lambda-1 & -2 \\ 2 & \lambda+1 \end{vmatrix} = 0 \Rightarrow \lambda^2 - 1 + 4 = 0$$

$$\Rightarrow \lambda^2 = -3$$

This has no real eigenvalues.

**Theorem 5.1.3** If  $A$  is an  $n \times n$  matrix, the following statements are equivalent.

- $\lambda$  is an eigenvalue of  $A$ .
- $\lambda$  is a solution of the characteristic equation  $\det(\lambda I - A) = 0$ .
- The system of equations  $(\lambda I - A)\mathbf{x} = \mathbf{0}$  has nontrivial solutions.
- There is a nonzero vector  $\mathbf{x}$  such that  $A\mathbf{x} = \lambda\mathbf{x}$ .

#9 find the characteristic equation, the eigenvalues, and bases for the eigenspaces of the matrix.

$$\begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix} \quad \begin{vmatrix} \lambda-6 & -3 & 8 \\ 0 & \lambda+2 & 0 \\ -1 & 0 & \lambda+3 \end{vmatrix} = 0$$

$$(\lambda+2) \begin{vmatrix} \lambda-6 & 8 \\ -1 & \lambda+3 \end{vmatrix} = 0 \Rightarrow (\lambda+2)(\lambda^2 - 3\lambda - 10) = 0$$

Characteristic equation  $\rightarrow (\lambda+2)^2(\lambda-5) = 0 \Rightarrow \lambda = -2, 5$  (eigenvalues)

$$\lambda_1 = -2 \text{ (alg. mult. 2)} \quad \begin{bmatrix} -8 & -3 & 8 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 = x_3 \\ x_2 = 0 \end{matrix}$$

basis for eigenspace is  $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ . (geom. mult. 1)

$$\lambda_2 = 5 \quad \begin{bmatrix} -1 & -3 & 8 \\ 0 & 7 & 0 \\ -1 & 0 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} x_1 = 8x_3 \\ x_2 = 0 \end{array}$$

basis :  $\left\{ \begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix} \right\}$

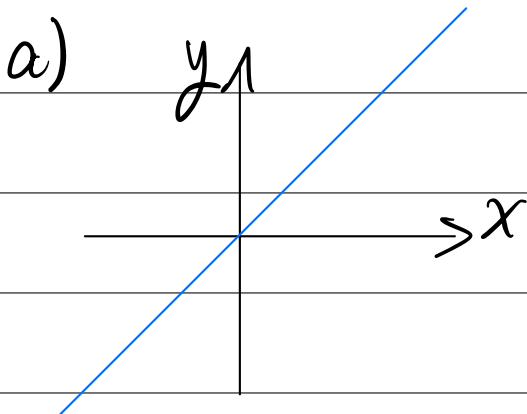
$\vec{x} \neq 0$  by definition

#19 In each part, find the eigenvalues and the corresponding eigenspaces of the stated matrix operator on  $\mathbb{R}^2$ . Use geometric reasoning to find the answers. No computations are needed.

$$A\vec{x} = \lambda \vec{x}$$

$\rightarrow$  parallel to  $\vec{x}$

- Reflection about the line  $y = x$ .
- Orthogonal projection onto the  $x$ -axis.
- Rotation about the origin through a positive angle of  $90^\circ$ .
- Contraction with factor  $k$  ( $0 \leq k < 1$ )
- Shear in the  $x$ -direction by a factor  $k$  ( $k \neq 0$ )

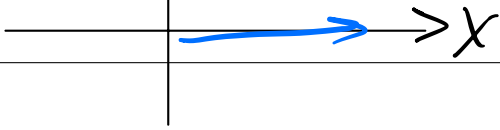


$$\lambda = 1, \text{ span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$\lambda = -1, \text{ span} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

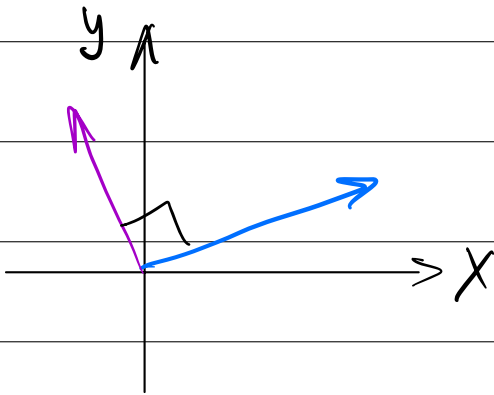
b)  $z$

$\lambda = 1$ ,  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$



c)

no real eigenvalues.



**Theorem 5.1.2** If  $A$  is an  $n \times n$  triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of  $A$  are the entries on the main diagonal of  $A$ .

**#14** Find the characteristic equation of the matrix by inspection.

$$\begin{bmatrix} 9 & -8 & 6 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

$$(\lambda - 9)(\lambda + 1)(\lambda - 3)(\lambda - 7) = 0$$

**#15** Find the eigenvalues and a basis for each eigenspace of the linear operator defined by the stated formula.

$$T(x, y) = (x + 4y, 2x + 3y) \rightarrow A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \text{ proceed as above}$$

Notes:  $\lambda = 0$  is an allowable eigenvalue  
 $\vec{x} = \vec{0}$  can't be an eigenvector by def.

**Theorem 5.1.4** A square matrix  $A$  is invertible if and only if  $\lambda = 0$  is not an eigenvalue of  $A$ .

Recall that  $\lambda$  is an eigenvalue of  $A$  if  
 $\exists$  a non zero vector  $x \ni A\vec{x} = \lambda\vec{x}$ .

Then  $\det(\lambda I - A) = 0$ .

$(\Rightarrow)$  We know  $\det(A) = 0 \Rightarrow \det(-A) = 0$

But if  $\lambda = 0$  is an eigenvalue of  $A$ ,  
then  $\det(\lambda I - A) = 0 \Rightarrow \det(-A) = 0$

$\Rightarrow A$  is not invertible.

$(\Leftarrow)$  If  $\lambda = 0$  is not an eigenvalue of  $A$ ,  
then there is no non zero vector  $\vec{x}$   
satisfying  $A\vec{x} = \vec{0} \Rightarrow$  the system

has only the trivial solution  
 $\Rightarrow A$  is invertible.

**Theorem 5.1.5** Equivalent Statements (extends Theorem 4.9.8)

If  $A$  is an  $n \times n$  matrix, then the following statements are equivalent.

- a)  $A$  is invertible.
- b)  $Ax = \mathbf{0}$  has only the trivial solution.
- c) The reduced row echelon form of  $A$  is  $I_n$ .
- d)  $A$  is expressible as a product of elementary matrices.
- e)  $Ax = \mathbf{b}$  is consistent for every  $n \times 1$  matrix  $\mathbf{b}$ .
- f)  $Ax = \mathbf{b}$  has exactly one solution for every  $n \times 1$  matrix  $\mathbf{b}$ .
- g)  $\det(A) \neq 0$ .
- h) The column vectors of  $A$  are distinct and linearly independent.
- i) The row vectors of  $A$  are distinct and linearly independent.
- j) The column vectors of  $A$  span  $R^n$ .
- k) The row vectors of  $A$  span  $R^n$ .
- l) The column vectors of  $A$  form a basis for  $R^n$ .
- m) The row vectors of  $A$  form a basis for  $R^n$ .
- n)  $A$  has rank  $n$ .
- o)  $A$  has nullity 0.
- p) The orthogonal complement of the null space of  $A$  is  $R^n$ .
- q) The orthogonal complement of the row space of  $A$  is  $\{\mathbf{0}\}$ .
- r)  $\lambda = 0$  is not an eigenvalue of  $A$ .